Exercise 5G: Solutions

1 ${}^7C_2 = 21, {}^6C_2 = 15$ and ${}^6C_1 = 6$. Clearly, ${}^7C_2 = {}^6C_2 + {}^6C_1$.

2 The $n = 7$ row is:

$$1 7 21 35 35 21 7 1$$

${}^7C_2 = 21$ since this is the third entry in the row.
${}^7C_4 = 35$ since this is the fifth entry in the row.

3 The $n = 8$ row is:

$$1 8 28 56 70 56 28 8 1$$

${}^8C_4 = 70$ since this is the fifth entry in the row.
${}^8C_6 = 28$ since this is the seventh entry in the row.

4 A set with 6 elements has $2^6 = 64$ subsets. Note that this includes the empty subset, which corresponds to selecting none of the DVDs.

5 A set of 5 elements has $2^5 = 32$ subsets.

6 A set with 10 elements has $2^{10} = 1024$ subsets.

7 A set with 6 elements has $2^6 - 1 = 63$ non-empty subsets.

8 A set with 8 elements has $2^8 - {}^8C_1 - {}^8C_0 = 256 - 8 - 1 = 247$ subsets with at least 2 elements.

9 If the set already contains the numbers 9 and 10, then we need to find the number of subsets of $\{1, 2, \ldots, 8\}$. There are $2^8 = 256$ of these.

10 Each subset of coins creates a different sum of money. We therefore need to find the number of non-empty subsets of a 4 element set. There are $2^4 - 1 = 15$ of these.

11a We consider the selfish subsets of size 1 through to 8. There is 1 selfish set of size 1, namely $\{1\}$.
If a selfish set has size 2, then it is of the form $\{2, a\}$ where a is chosen from the remaining 7 numbers. This can be done in ${}^7C_1$ ways.
If a selfish set has size 3, then it is of the form $\{3, a, b\}$ where the two numbers a and b are chosen from the remaining 7 numbers. This can be done in ${}^7C_2$ ways.
Continuing in this fashion, we find that the number of selfish sets is just the sum of entries in row $n = 7$ of Pascal's Triangle. Therefore, there are $2^7 = 128$ selfish sets.

b We consider the selfish subsets of size 1 through to 8.
There is 1 selfish subset of size 1. Its compliment is also selfish, as it has 7 elements and contains the number 7.
A selfish set of size 2 is of the form $\{2, a\}$, where $a \neq 2$. Since the compliment is also selfish, $a \neq 6$. Therefore, a can be chosen from the remaining 6 numbers. This can be done in ${}^6C_1$ ways.
A selfish set of size 3 is of the form $\{3, a, b\}$, where $a, b \neq 3$. Since the compliment is also selfish, $a, b \neq 5$. Therefore, a and b can be chosen from the remaining 6 numbers. This can be done in ${}^6C_2$ ways.
A selfish set of size 4 is of the form $\{4, a, b, c\}$, where $a, b, c \neq 4$. The compliment cannot also be selfish, since the compliment has 4 elements but does not contain the number 4.
A selfish set of size 5 is of the form $\{5, a, b, c, d\}$, where $a, b, c, d \neq 5$. Since the compliment is also selfish, $a, b, c, d \neq 3$. Therefore, a, b, c, d can be chosen from the remaining 6 numbers. This can be done in ${}^6C_4$ ways.
Continuing in this fashion, we find that the number of selfish sets with a selfish compliment is just the sum of entries in row $n = 6$ of Pascal's Triangle, less ${}^6C_3$. Therefore, there are $2^6 - {}^6C_3 = 44$ selfish sets whose compliment is also selfish.